Non-Markovian disentanglement dynamics of two-qubit system

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Abstract

We investigated the disentanglement dynamics of two-qubit system in Non-Markovian approach. We showed that only the couple strength with the environment near to or less than fine-structure constant 1/137, entanglement appear exponential decay for a certain class of two-qubit entangled state. While the coupling between qubit and the environment is much larger, system always appears the sudden-death of entanglement even in the vacuum environment.

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I. INTRODUCTION

A multipartite quantum system, in addition to local quantum coherence that exists within each of subsystems, may have nonlocal or distributed quantum coherence that exists among several distinct subsystems. This property is entangle, which is superposition of the internal states of the systems and cannot be separated into product states of the individual subsystem. It is recognized as entirely quantum-mechanical effect and have played a crucial role in practical application ranging from quantum information[1, 2], cryptography[3] and quantum computation[4, 5], to atomic and molecular spectroscopy[6, 7].

Recent, many groups were able to prepare entangled states in a variety of physical systems and experimental setups, demonstrating an impressive ability to manipulate and detect them efficiently.[8, 9, 10, 11, 12, 13, 14, 15] In particular, Almeida *et al.*[15] showed that, using an all-optical experimental setup, even when the environment-induced decay of each system is asymptotic, quantum entanglement may appear "entanglement sudden death", called ESD[18], that is, entanglement terminates completely after a finite interval, without a smoothly diminishing long-time tail.

As we known, a major obstacle for the controlled entanglement of more and more subsystems remains with the capacity of achieving perfect screening of the system from the environment. After some time, the unavoidable residual interaction with the environment induces mixing of the system state, and thus the emergence of classical correlations at the expense of quantum entanglement. Hence, we face the high relevant task of understanding the sources of entanglement decay, what implies the identification of the associated time scales. In addition to be enlightened by the experimental discovery of ESD, a large number of theoretical literature have investigated the disentanglement dynamics.[16, 17, 18, 19] Z. Ficek and R. Tanaś [16] propose the review with an overview of the mathematical apparatus necessary for describing the interaction of atoms with the electromagnetic field. Present the master equation technique and describe a more general formalism based on the quantum jump approach. F. Mintert *et al.*[17] start with a short recollection of environment models adapted for decoherence process in a typical quantum optical context under the assumption of complete positivity and Markovian dynamics in the Lindblad form. Yu and Eberly [18, 19] showed that the dynamics of the quantum entanglement between two qubits interacting independently with either quantum noise or classical noise displays a completely different

behavior from the dynamics of the local coherence. Instead of the exponential decay in time of the local coherence, quantum entanglement may disappear within a finite time in the dynamical evolution. The "entanglement sudden death" has been experimentally demonstrated by Almeida. However, it is surprising that few if any fundamental treatment exist of decoherence that include the dynamics of disentanglement on better than an Markovian approximation or phenomenological. Although, the use of the Markovian approximation is justified in a large variety of quantum optical experiments where entanglement has been produced, one should notice that Non-Markovian effects are important in the description of some condensed-matter system[20], such as the quantum dot qubit(s) system. Therefore, a Non-Markovian effects of the decoherence in any viable realization of qubits is desirable.

In this paper we examined the disentanglement dynamics of two entangled qubits due to spontaneous emission, where the interaction with the environment without rotating-wave approximation and the treatment process without Markovian approximation. It is found that disentanglement always take only a finite-time to be completed, called "entanglement sudden death", when the coupling between qubit and environment is strong. While the coupling with dissipation environments is weak to fine structure constant $1/137$, the disentanglement change from exponential decay to entanglement sudden-death with the increasing of the portion of the double excitation component in the initial entangled state. We describe the sudden-death time of entanglement or the realized lifetime of the given two-qubit entanglement system through the measurable parameters: coupling constant with the enviornment α, energy splitting Δ and cut-off frequency ω_c. If we consider entanglements as the central resource of most types of quantum information processing, it is the most relevant question in entanglement experiment under the environment-induced mixing.

The paper is organized as follows: In sec. II we introduce the Hamiltonian without rotating-wave approximation in the two-qubit environment interaction and solve it in terms of Non-Markovian treatment. The dependence of the concurrence on the different initial condition and the coupling strength to the dissipation environment, are discussed in sec. III. Finally, the conclusion is given in sec. IV.

II. THE MODEL AND THEORY

This paper is concerned primarily with two two-level systems, since it is generally believed that entanglement of only two microscopic quantum systems (qubits, atoms) is essential to implement quantum protocols such as quantum computation. We consider two two-level subsystem A, B and assume that each subsystem interacts independently with the environment, a well justified assumption wherever the particles composing your system are sufficiently separated from each other, and therefore, no collective environment effects must be taken into account. In non-rotating wave form such a model may be formulated to the following total Hamiltonian (set $\hbar = 1$):

$$H(t) = H_{qu} + H_{env} + H_{int}, \tag{1}$$

with

$$H_{qu} = -\frac{1}{2}\Delta_A\sigma_z^A - \frac{1}{2}\Delta_B\sigma_z^B, \tag{2}$$

$$H_{env} = \sum_k \omega_k a_k^+ a_k + \sum_k \nu_k b_k^+ b_k, \tag{3}$$

$$H_{int} = \frac{1}{2}\sum_k g_k(a_k^+ + a_k)\sigma_x^A + \frac{1}{2}\sum_k f_k(b_k^+ + b_k)\sigma_x^B, \tag{4}$$

where the Hamiltonian of the two qubits H_{qu}, the two independently environments H_{env}, the interaction H_{int}. Here σ_i $(i = x, y, z)$ denotes the usually Pauli spin matrices, Δ_A (Δ_B) describes the energy splitting in the A (B) qubit. a_k^+ (b_k^+), a_k (b_k) and ω_k (ν_k) are the creation, annihilation operator and energy with wave vector k in the A (B) qubit environment. g_k and f_k are the qubit-environment coupling strength. Yu and Eberly *etc.*[18, 19] has employed the similar model, but the rotating-wave approximation is valid. Two environments are completely defined by the spectral density:

$$J(\omega) = \sum_k g_k^2 \delta(\omega - \omega_k). \tag{5}$$

We consider the Ohmic bath $J(\omega) = 2\alpha\omega\theta(\omega_c - \omega)$ in this work, where α is the dimensionless coupling constant and $\theta(x)$ is the usual step function.

In order to simplify the non rotating-wave term, we apply a canonical transformation, $H' = \exp(s)H\exp(-s)$ with the generator[21]:

$$S = \sum_k \frac{g_k}{2\omega_k}\xi_k^A(a_k^+ - a_k)\sigma_x^A + \sum_k \frac{f_k}{2\nu_k}\xi_k^B(b_k^+ - b_k)\sigma_x^B. \tag{6}$$

Then decompose the transformed Hamiltonian H' into three parts:

$$H^{'} = H_0^{'} + H_1^{'} + H_2^{'}, \tag{7}$$

where the three parts include the analogous form for A and B qubit,

$$H_0^{'} = H_{0A}^{'} + H_{0B}^{'} \tag{8}$$

with

$$H_{0A}^{'} = -\frac{1}{2}\eta^A \Delta_A \sigma_z^A + \sum_k \omega_k a_k^+ a_k - \sum_k \frac{g_k^2}{4\omega_k}\xi_k^A(2-\xi_k^A), \tag{9}$$

$$H_{0B}^{'} = -\frac{1}{2}\eta^B \Delta_B \sigma_z^B + \sum_k \nu_k b_k^+ b_k - \sum_k \frac{f_k^2}{4\nu_k}\xi_k^B(2-\xi_k^B). \tag{10}$$

As the same style, $H_1^{'} = H_{1A}^{'} + H_{1B}^{'}$ and $H_2^{'} = H_{2A}^{'} + H_{2B}^{'}$, where

$$H_{1A}^{'} = \frac{1}{2}\sum_k \eta^A \Delta_A \frac{g_k \xi_k^A}{\omega_k}(a_k^+ \sigma_-^A + a_k^- \sigma_+^A), \tag{11}$$

$$\begin{aligned} H_{2A}^{'} \; = \; & -\frac{1}{2}\Delta\sigma_x \left[\cosh(\sum_k \frac{g_k}{\omega_k}\xi_k(b_k^+ - b_k)) - \eta\right] \\ & -i\frac{\Delta}{2}\sigma_y \left[\sinh(\sum_k \frac{g_k}{\omega_k}\xi_k(b_k^+ - b_k)) - \eta \sum_k \frac{g_k}{\omega_k}\xi_k(b_k^+ - b_k)\right], \end{aligned} \tag{12}$$

with

$$\eta^A = \exp[-\sum_k \frac{g_k^2}{2\omega_k^2}(\xi_k^A)^2], \eta^B = \exp[-\sum_k \frac{f_k^2}{2\nu_k^2}(\xi_k^B)^2] \tag{13}$$

$$\xi_k^A = \frac{\omega_k}{\omega_k + \eta^A \Delta_A}, \xi_k^B = \frac{\nu_k}{\nu_k + \eta^B \Delta_B}. \tag{14}$$

Here $\sigma_\pm^A = \sigma_x^A \mp \sigma_y^A$, $H_0^{'}$ is the Hamiltonian of the noninteracting qubits and environment, $H_1^{'}$ and $H_2^{'}$ are the interaction Hamiltonian in increasing order of the qubit-environment coupling strength g_k and f_k. Comparing H_1 to $H_1^{'}$, the term of H_1 is replaced by the similar rotating-wave approximation term in $H_1^{'}$, while the qubit-environment coupling strength g_k in H_1 is replaced by $g_k\eta^A\Delta_A/(\omega_k + \eta^A\Delta_A)$ in $H_1^{'}$. As we seen, $g_k\eta^A\Delta_A/(\omega_k + \eta^A\Delta_A) < g_k$, that is to say, the counter-rotating terms decrease the coupling strength with the environment.

Approximately to the order of g_k^2 and f_k^2, we write the total Hamiltonian as $H^{'} = H_0^{'}+H_1^{'}$. In the interaction picture,

$$
\begin{aligned}
V_I^{'}(t) \;=\; & \sum_k \eta^A \Delta_A \frac{g_k \xi_k^A}{\omega_k} a_k^{+} \sigma_-^A \exp\left[i(\omega_k - \eta^A \Delta_A)t\right] \\
& + \sum_k \eta^A \Delta_A \frac{g_k \xi_k^A}{\omega_k} a_k \sigma_+^A \exp\left[-i(\omega_k - \eta^A \Delta_A)t\right] \\
& + \sum_k \eta^B \Delta_B \frac{f_k \xi_k^B}{\nu_k} b_k^{+} \sigma_-^B \exp\left[i(\nu_k - \eta^B \Delta_B)t\right] \\
& + \sum_k \eta^B \Delta_B \frac{f_k \xi_k^B}{\nu_k} b_k \sigma_+^B \exp\left[-i(\nu_k - \eta^B \Delta_B)t\right].
\end{aligned} \tag{15}
$$

We consider in general a system denoted by S interacting with a reservoir or environment denoted by R. The combined density operator is denoted by ρ_{SR}. The reduced density operator for the system ρ_S is obtained by taking a trace over the reservoir coordinates, i.e., $\rho_S = Tr_R(\rho_{SR})$. The equation of motion for ρ_{SR} is given by

$$\frac{d}{dt}\rho_{SR}(t) = -i[V_I(t), \rho_{SR}]. \tag{16}$$

After S transformation,

$$\frac{d}{dt}\rho_{SR}^{'}(t) = -i[V_I^{'}(t), \rho_{SR}^{'}]. \tag{17}$$

This equation can be formally integrated, and we obtain

$$\rho_{SR}^{'}(t) = \rho_{SR}^{'}(t_i) - i\int_{t_i}^{t} [V_I^{'}(t^{'}), \rho_{SR}^{'}(t^{'})]dt^{'}. \tag{18}$$

Here t_i is an initial time when the interaction starts, supposing $t_i = 0$. On substituting $\rho_{SR}^{'}(t)$ into Eq.(18), we find the equation of motion

$$\frac{d}{dt}\rho_{SR}^{'}(t) = -i[V_I^{'}(t), \rho_{SR}^{'}(0)] - \int_0^t [V_I^{'}(t), [V_I^{'}(t^{'}), \rho_{SR}^{'}(t^{'})]]dt^{'}. \tag{19}$$

We now employ the Born approximation[16, 17, 22] in which the interaction between the qubit system and the environment is suppose to be weak, and there is no back reaction effect of the qubits on the enviornment. In this approximation, the state of the environment does not change in time, and we can write the density operator $\rho_{SR}^{'}(t)$ as $\rho_{SR}^{'}(t) = \rho_S^{'}(t)\rho_R^{'}(0)$. Under this approximation, Eq.(19) simplifies to

$$\frac{d}{dt}\rho_S^{'}(t)\rho_R^{'}(0) = -i[V_I^{'}(t), \rho_S^{'}(0)\rho_R^{'}(0)] - \int_0^t [V_I^{'}(t), [V_I^{'}(t^{'}), \rho_S^{'}(t^{'})\rho_R(0)]]dt^{'}. \tag{20}$$

Substituting $V_I^{'}(t)$ into Eq.(20) and assuming that the environment modes is in thermalization, the Tr_R are given by:

$$Tr_R[b_k^+ b_k \rho_R] = Tr_R[b_k \rho_R b_k^+] = n_k, \tag{21}$$

$$Tr_R[b_k b_k^+ \rho_R] = Tr_R[b_k^+ \rho_R b_k] = n_k + 1. \tag{22}$$

Then,

$$\begin{aligned}
&\frac{d}{dt}\rho_S^{'}(t) \qquad\qquad (23)\\
= &-\int_0^t \sum_k (\eta^A \Delta_A \frac{g_k \xi_k^A}{\omega_k})^2 n_k^A [\sigma_-^A \sigma_+^A \rho_S^{'I}(t^{'}) - \sigma_+^A \rho_S^{'I}(t^{'}) \sigma_-^A] \exp\left[i(\omega_k - \eta^A \Delta_A)(t - t^{'})\right] dt^{'}\\
&-\int_0^t \sum_k (\eta^A \Delta_A \frac{g_k \xi_k^A}{\omega_k})^2 (n_k^A + 1) [\rho_S^{'I}(t^{'}) \sigma_+^A \sigma_-^A - \sigma_-^A \rho_S^{'I}(t^{'}) \sigma_+^A] \exp\left[i(\omega_k - \eta^A \Delta_A)(t - t^{'})\right] dt^{'}\\
&-\int_0^t \sum_k (\eta^A \Delta_A \frac{g_k \xi_k^A}{\omega_k})^2 n_k^A [\rho_S^{'I}(t^{'}) \sigma_-^A \sigma_+^A - \sigma_+^A \rho_S^{'I}(t^{'}) \sigma_-^A] \exp\left[-i(\omega_k - \eta^A \Delta_A)(t - t^{'})\right] dt^{'}\\
&-\int_0^t \sum_k (\eta^A \Delta_A \frac{g_k \xi_k^A}{\omega_k})^2 (n_k^A + 1) [\sigma_+^A \sigma_-^A \rho_S^{'I}(t^{'}) - \sigma_-^A \rho_S^{'I}(t^{'}) \sigma_+^A] \exp\left[-i(\omega_k - \eta^A \Delta_A)(t - t^{'})\right] dt^{'}\\
&-\int_0^t \sum_k (\eta^B \Delta_B \frac{f_k \xi_k^B}{\nu_k})^2 n_k^B [\sigma_-^B \sigma_+^B \rho_S^{'I}(t^{'}) - \sigma_+^B \rho_S^{'I}(t^{'}) \sigma_-^B] \exp\left[i(\nu_k - \eta^B \Delta_B)(t - t^{'})\right] dt^{'}\\
&-\int_0^t \sum_k (\eta^B \Delta_B \frac{f_k \xi_k^B}{\nu_k})^2 (n_k^B + 1) [\rho_S^{'I}(t^{'}) \sigma_+^B \sigma_-^B - \sigma_-^B \rho_S^{'I}(t^{'}) \sigma_+^B] \exp\left[i(\nu_k - \eta^B \Delta_B)(t - t^{'})\right] dt^{'}\\
&-\int_0^t \sum_k (\eta^B \Delta_B \frac{f_k \xi_k^B}{\nu_k})^2 n_k^B [\rho_S^{'I}(t^{'}) \sigma_-^B \sigma_+^B - \sigma_+^B \rho_S^{'I}(t^{'}) \sigma_-^B] \exp\left[-i(\nu_k - \eta^B \Delta_B)(t - t^{'})\right] dt^{'}\\
&-\int_0^t \sum_k (\eta^B \Delta_B \frac{f_k \xi_k^B}{\nu_k})^2 (n_k^B + 1) [\sigma_+^B \sigma_-^B \rho_S^{'I}(t^{'}) - \sigma_-^B \rho_S^{'I}(t^{'}) \sigma_+^B] \exp\left[-i(\nu_k - \eta^B \Delta_B)(t - t^{'})\right] dt^{'}.
\end{aligned}$$

In this equation, the n_k and $n_k + 1$ term on the right hand side describe, respectively, decay and excitation process, with rate which depend on the temperature, here parameterized by n_k, the average thermal excitation of the reservoir. In this work, we study the limit of zero temperature, $n_k = 0$, that is to say only the spontaneous decay term survives leading to purely dissipative process.

The matrix equation is solved in the representation spanned by the standard two-qubit

product states basis $|1\rangle = |\uparrow\uparrow\rangle, |2\rangle = |\uparrow\downarrow\rangle, |3\rangle = |\downarrow\uparrow\rangle, |4\rangle = |\downarrow\downarrow\rangle$. After Laplace transformation and convolution theorem, the master equation of the system of two qubits can be obtained as follow[23]:

$$\begin{aligned}P\overline{\rho_S^{'}(P)} - \rho_S^{'}(0) \;=\; &-\sum_k \frac{(\eta^A\Delta_A\frac{g_k\xi_k^A}{\omega_k})^2}{P-i(\omega_k-\eta^A\Delta_A)}[\overline{\rho_S^{'}(P)}\sigma_+^A\sigma_-^A - \sigma_-^A\overline{\rho_S^{'}(P)}\sigma_+^A] \\ &-\sum_k \frac{(\eta^B\Delta_B\frac{g_k\xi_k^B}{\nu_k})^2}{P-i(\nu_k-\eta^B\Delta_B)}[\overline{\rho_S^{'}(P)}\sigma_+^B\sigma_-^B - \sigma_-^B\overline{\rho_S^{'}(P)}\sigma_+^B] \\ &-\sum_k \frac{(\eta^A\Delta_A\frac{g_k\xi_k^A}{\omega_k})^2}{P+i(\omega_k-\eta^A\Delta_A)}[\sigma_+^A\sigma_-^A\overline{\rho_S^{'}(P)} - \sigma_-^A\overline{\rho_S^{'}(P)}\sigma_+^A] \\ &-\sum_k \frac{(\eta^B\Delta_B\frac{g_k\xi_k^B}{\nu_k})^2}{P+i(\nu_k-\eta^B\Delta_B)}[\sigma_+^B\sigma_-^B\overline{\rho_S^{'}(P)} - \sigma_-^B\overline{\rho_S^{'}(P)}\sigma_+^B].\end{aligned} \tag{24}$$

Denote the summation of the environment degree of freedom $A_+ = \sum_k \frac{(\eta^A\Delta_A\frac{g_k\xi_k^A}{\omega_k})^2}{P+i(\omega_k-\eta^A\Delta_A)}$, $A_- = \sum_k \frac{(\eta^A\Delta_A\frac{g_k\xi_k^A}{\omega_k})^2}{P-i(\omega_k-\eta^A\Delta_A)}$, $B_+ = \sum_k \frac{(\eta^B\Delta_B\frac{g_k\xi_k^B}{\nu_k})^2}{P+i(\nu_k-\eta^B\Delta_B)}$and $B_- = \sum_k \frac{(\eta^B\Delta_B\frac{g_k\xi_k^B}{\nu_k})^2}{P-i(\nu_k-\eta^B\Delta_B)}$. The decay rate is dependent on the process, as seen from A_+, A_-, B_+ and B_-, instead of constant for all process in Markovian approximation. We shall therefore focus on the precise time scales of every decay process.

According to the Kronecker product property and technique to Lyapunov matrix equation in matrix theory, expand matrix into vector along row of the matrix from two sides of master equation,

$$\begin{aligned}&\Big\{PI_{16\times16} + [A_-I_{4\times4}\otimes(\sigma_+^A\sigma_-^A\otimes I_{2\times2})^T + B_-I_{4\times4}\otimes(I_{2\times2}\otimes\sigma_+^B\sigma_-^B)^T] \\ &-[A_-(\sigma_-^A\otimes I_{2\times2})\otimes(\sigma_+^A\otimes I_{2\times2})^T + B_-(I_{2\times2}\otimes\sigma_-^B)\otimes(I_{2\times2}\otimes\sigma_+^B)^T] \\ &-[A_+(\sigma_-^A\otimes I_{2\times2})\otimes(\sigma_+^A\otimes I_{2\times2})^T + B_+(I_{2\times2}\otimes\sigma_-^B)\otimes(I_{2\times2}\otimes\sigma_+^B)^T] \\ &+[A_+(\sigma_+^A\sigma_-^A\otimes I_{2\times2})\otimes I_{4\times4} + B_-(I_{2\times2}\otimes\sigma_+^B\sigma_-^B)\otimes I_{4\times4}]\Big\}Vec[\overline{\rho_S^{'}(P)}] = Vec[\rho_S^{'}(0)].\end{aligned} \tag{25}$$

The 4×4 matrix equation transformed into 16×16 matrix equation with the form

$$U(P)_{16\times16}Vec[\overline{\rho_S^{'}(P)}] = Vec[\rho_S^{'}(0)] \tag{26}$$

where $Vec[\overline{\rho_S^{'}(P)}]$ is the vector of row expanding of matrix $\overline{\rho_S^{'}(P)}$. The solution formally is

$$Vec[\overline{\rho_S^{'}(P)}] = U(P)^{-1}_{16\times16}Vec[\rho_S^{'}(0)]. \tag{27}$$

Inverse Laplace transformation to time parameter space,

$$\mathscr{L}^{-1}Vec[\overline{\rho_S^{'}(P)}] = \mathscr{L}^{-1}U(P)^{-1}_{16\times 16}Vec[\rho_S^{'}(0)]. \tag{28}$$

i.e.

$$Vec[\rho_S^{'I}(t)] = \mathscr{L}^{-1}U(P)^{-1}_{16\times 16}Vec[[\rho_S^{'}(0)]]. \tag{29}$$

$\mathscr{L}^{-1}U(P)^{-1}_{16\times 16}$ can be obtained (see Appendix).

Compared with Markovian approximation, decoherence rates $\gamma(\omega)$ in our results becomes frequency dependent. Due to entanglement and environment interaction together, the decay rate for variety process are different, some increase slower, some increase faster, i.e. the two bathes has indirect interaction through the two entangled qubits. That is more general and physical.

Therefore, the reduced density matrix $\rho_S^{'}(t)$ in the Schrodinger picture is obtained $\rho_S^{'}(t) = \exp(-iH_0^{'}t)\rho_S^{'I}(t)\exp(iH_0^{'}t)$, the matrix form is

$$\begin{aligned}
\rho_S^{'}(t) &= \left[\begin{pmatrix} \exp(i\frac{\eta^A\Delta_A}{2}t) & 0 \\ 0 & \exp(-i\frac{\eta^A\Delta_A}{2}t) \end{pmatrix} \otimes \begin{pmatrix} \exp(i\frac{\eta^B\Delta_B}{2}t) & 0 \\ 0 & \exp(-i\frac{\eta^B\Delta_B}{2}t) \end{pmatrix}\right] \\
&\quad \rho_S^{'I}(t)\left[\begin{pmatrix} \exp(-i\frac{\eta^A\Delta_A}{2}t) & 0 \\ 0 & \exp(i\frac{\eta^A\Delta_A}{2}t) \end{pmatrix} \otimes \begin{pmatrix} \exp(-i\frac{\eta^B\Delta_B}{2}t) & 0 \\ 0 & \exp(i\frac{\eta^B\Delta_B}{2}t) \end{pmatrix}\right].
\end{aligned} \tag{30}$$

Transform $\rho_S^{'}(t)$ into $\rho_S(t)$ through $\rho_S(t) = Tr_R[\exp(-S)\rho_S^{'}(t)\rho_R(0)\exp(S)]$, denoting $X_A = \sum_k \frac{g_k}{2\omega_k}\xi_k^A(a_k^+ - a_k)$, $X_B = \sum_k \frac{f_k}{2\nu_k}\xi_k^B(b_k^+ - b_k)$, so

$$\begin{aligned}
\rho_S(t) &= Tr_R[(\cosh X_A - \sinh X_A\sigma_x^A)\otimes(\cosh X_B - \sinh X_B\sigma_x^B)\rho_S^{'}(t)\rho_R \\
&\quad (\cosh X_A + \sinh X_A\sigma_x^A)\otimes(\cosh X_B + \sinh X_B\sigma_x^B) \\
&= \frac{1+\eta^A}{2}\frac{1+\eta^B}{2}\rho_S^{'}(t) + \frac{1+\eta^A}{2}\frac{1-\eta^B}{2}(I_{2\times 2}\otimes\sigma_x^B)\rho_S^{'}(t)(I_{2\times 2}\otimes\sigma_x^B) \\
&\quad +\frac{1-\eta^A}{2}\frac{1+\eta^B}{2}(\sigma_x^A\otimes I_{2\times 2})\rho_S^{'}(t)(\sigma_x^A\otimes I_{2\times 2}) \\
&\quad +\frac{1-\eta^A}{2}\frac{1-\eta^B}{2}(\sigma_x^A\otimes\sigma_x^B)\rho_S^{'}(t)(\sigma_x^A\otimes\sigma_x^B).
\end{aligned} \tag{31}$$

Until now, we obtain the reduced density matrix in all kinds of initial state.

Although a general solution to this problem, for arbitrary system dynamics and decoherence mechanisms is still out of reach, out technical machinery, developed in the previous section allows to treat arguably all situations encountered in typical state of the art experiments, as in the quantum optics and condensed matter.

III. THE RESULT AND DISCUSSION

We assume that at t=0, the two qubits and environment are described by the product state $\exp(-S)\rho_S(0)\rho_R(0)\exp(S) = \Psi_{AB} \otimes |0\rangle_A|0\rangle_B$, where Ψ_{AB} is the entangled initial state of the two qubits and $|0\rangle_A|0\rangle_B$ is the vacuum state of two environments. Let us assume that the initial density matrix is only practically coherence of a familiar type (one of the atoms is excited, but it is not certain which one). This is easily expressed in the following form[19, 24]

$$\rho_S^{'}(0) = \frac{1}{3}\begin{pmatrix} d & 0 & 0 & 0 \\ 0 & c & z & 0 \\ 0 & z^* & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}. \tag{32}$$

where the factor 1/3 is for notational convenience. In order to compare with previous results, consider an important class of mixed state with single parameter a satisfying initially $a \geq 0$, $d = 1 - a$, and $b = c = z = 1$. We will use Wootter's concurrence to quantify the degree of entanglement[25, 26]. Let ρ be density matrix of the pair of qubits expressed in the standard basis. The concurrence may be calculated explicitly from the density matrix ρ for qubits A and B: $C = \max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4})$, where the quantities λ_i are the eigenvalues of the matrix M: $M = \rho(\sigma_y^A \otimes \sigma_y^B)\rho^*(\sigma_y^A \otimes \sigma_y^B)$, arranged in decreasing order. Here ρ^* denotes the complex conjugation of ρ in the standard basis. It can be shown that the concurrence varies from 0 for a disentangled state to $C = 1$ for a maximally entangled state.

Firstly consider very weak qubit-environment interaction, $\alpha_A = \alpha_B = 0.01$, which is larger a bit than the fine-structure constant 1/137. Here and in the following, energies Δ_A, and Δ_B are expressed in units of ω_c, times in units of ω_c^{-1}. We assume $\Delta_A = 0.2$, $\Delta_B = 0.4$. In Fig.1, the time evolution of the concurrence for various values of the parameter a is shown. The figure shows that for all a values almost between 0.3 and 1, concurrence decays is completed in a finite-time, which is the effect of "entanglement sudden death" [15, 18], but for smaller a's the time for completed decay is infinite, which is consistent with Ref.15 and 19. The result indicated that in the weak dissipation environment, such as the all-optical setup in Ref.15, the Markovian approximation and rotating-wave approximation are available. When the coupling constant to the enviornment is near to or less than fine-structure constant, we see that the quantum dissipation of the vacuum environment is

not sufficient to completely destroy the entanglement in a finite time in some situations. The sudden death of entanglement results from the decays of the mixed double excitation state component. With increasing of the mixed double excitation state component, a value, concurrence change from exponential decay to sudden death. The entanglement has another unusual relaxation property: different entangled states, corresponding to different values of a, with the same initial degrees of entanglement may evolve with different route, some showing entanglement sudden death, some not, some decay faster, some slower. That is to say, we can prepare certain initial state to prolong entanglement time.

Next, consider large qubit-environment interaction, $\alpha_A = \alpha_B = 0.05$, the other parameters and initial entangled state are same with Fig.1. The time evolution of the concurrence through the entire range of different a values is plotted in Fig.2. As we shown, concurrence actually goes abruptly to zero in a finite time and remains zero thereafter. That is to say the entanglement sudden-death always happens. In the first example above, we have shown that the entanglement can last for infinite period in the vacuum reservoir for some initial entangle state. However, in Fig.2 the sudden death of entanglement always happens no matter which entangled state the qubit are initially in. That is also shown that the disentanglement dynamics varies with the coupling strength with the enviornment or the rotating-wave approximation and Markovian approximation is unavailable, when the coupling to the environment is much larger than the fine-structure constant. Fig.3 shows concurrence for $\alpha_A = \alpha_B = 0.1$, under the same initial condition. It is observed that in the same initial state, the death time decreases as the increasing of the strength of qubit-environment interaction.

IV. CONCLUSION

In this paper, we consider two two-level qubits that are spatially separated from each other and independently coupled to local vacuum environments. We investigated the dynamics evolution of entanglement between the qubits. We show that, for a certain class of two-qubit entangled state, the entanglement measured by concurrence can change from exponential decay to sudden death with increasing of the mixed double excitation state component in the case of weak coupling with environment. Increasing coupling strength, the entanglement sudden-death always happens no matter which entangled state the qubit are initially in. The

exponential decay of entanglement is a very special result to the weak dissipation vacuum reservoir. The entangle sudden death time in our result is obtained from the physical parameter: coupling constant α, energy splitting Δ_A, Δ_B and cut-off frequency ω_c. Finally, we hope that this work will stimulate more experimental and theoretical works in quantum information and computation for quantum optical control.

Acknowledgement

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Appendix

In this Appendix, we give details of how to inverse Laplace transformation to time parameter space. $U(P)^{-1}_{16\times 16}$ is composed by the matrix element: $\frac{1}{P}$, $\frac{1}{P+A_-}$, $\frac{1}{P+A_+}$, $\frac{1}{P+A_-+A_+}$, $\frac{1}{P+A_-+B_+}$, $\frac{1}{P+A_-+A_++B_+}$, $\frac{1}{P+A_-+A_++B_++B_-}$ *etc.* Then $\mathscr{L}^{-1}U(P)^{-1}_{16\times 16}$ is inverse every matrix element. As we know, $\mathscr{L}^{-1}\frac{1}{P}=1$. Solve $\mathscr{L}^{-1}\frac{1}{P+A_-}$ *etc.* through the following method.

$$\mathscr{L}^{-1}\frac{1}{P+A_-}=\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\frac{\exp(Pt)}{P+\sum_k\frac{(\eta^A\Delta_A\frac{g_k\xi_k^A}{\omega_k})^2}{P-i(\omega_k-\eta^A\Delta_A)}}dP \tag{A1}$$

Then Changing P to $i\omega+0^+$,[27]

$$\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\frac{\exp(Pt)}{P+\sum_k\frac{(\eta^A\Delta_A\frac{g_k\xi_k^A}{\omega_k})^2}{P-i(\omega_k-\eta^A\Delta_A)}}dP=\frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty}\frac{\exp(i\omega t+0^+)}{\omega-\sum_k\frac{(\eta^A\Delta_A\frac{g_k\xi_k^A}{\omega_k})^2}{(\omega+\eta^A\Delta_A)-\omega_k-i0^+}}d\omega. \tag{A2}$$

Denote $R(\omega)$ and $\gamma(\omega)$ as the real and imaginary parts of $\sum_k(\eta^A\Delta_A\frac{g_k\xi_k^A}{\omega_k})^2/(\omega-\omega_k-i0^+)$,

$$\begin{aligned} R(\omega) &= \sum_k\wp\frac{(\eta^A\Delta_A\frac{g_k\xi_k^A}{\omega_k})^2}{\omega-\omega_k}=(\eta\Delta)^2\wp\int_0^\infty d\omega'\frac{J(\omega')}{(\omega-\omega')(\omega'+\eta\Delta)^2} \\ &= -2\alpha\frac{(\eta\Delta)^2}{\omega+\eta\Delta}\left\{\frac{\omega_c}{\omega_c+\eta\Delta}-\frac{\omega}{\omega+\eta\Delta}\ln\left[\frac{|\omega|\,(\omega_c+\eta\Delta)}{\eta\Delta(\omega_c-\omega)}\right]\right\}, \end{aligned} \tag{A3}$$

and

$$\gamma(\omega) = \pi \sum_k (\eta^A \Delta_A \frac{g_k \xi_k^A}{\omega_k})^2 \delta(\omega - \omega_k) = \pi(\eta\Delta)^2 \frac{J(\omega)}{(\omega + \eta\Delta)^2}$$
$$= 2\alpha\pi\omega \frac{(\eta\Delta)^2}{(\omega + \eta\Delta)^2}. \tag{A4}$$

Where $\wp$ stands for Cauchy principal value.

$$\mathscr{L}^{-1} \frac{1}{P + A_-} = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\exp(i\omega t + 0^+)}{\omega - R(\omega + \eta^A \Delta_A) + i\gamma(\omega + \eta^A \Delta_A)} d\omega \tag{A5}$$
$$= \exp[i\omega_{01} t - \gamma(\omega_{01} + \eta^A \Delta_A) t]$$

where ω_{01} is the solution of equation $\omega - R(\omega + \eta^A \Delta_A) = 0$ and is the Lamb shift due to the local interaction of the qubit with the enviornment.

In the same way,

$$\mathscr{L}^{-1} \frac{1}{P + A_+} = \exp[-i\omega_{01} t - \gamma(\omega_{01} + \eta^A \Delta_A) t]. \tag{A6}$$

It is clear that $\mathscr{L}^{-1} \frac{1}{P+A_-}$ conjugate with $\mathscr{L}^{-1} \frac{1}{P+A_+}$.

$$\mathscr{L}^{-1} \frac{1}{P + A_- + A_+} = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\exp(i\omega t + 0^+)}{\omega + i\gamma(\eta^A \Delta_A) + i\gamma(\eta^A \Delta_A)} d\omega \tag{A7}$$
$$= \exp[-2\gamma(\eta^A \Delta_A) t], \tag{A8}$$

The decay for $\mathscr{L}^{-1} \frac{1}{P+A_-+A_+}$ accelerated (by a factor of almost two) as compared to $\mathscr{L}^{-1} \frac{1}{P+A_+}$, under the influence of zero temperature environment.[17]

$$\mathscr{L}^{-1} \frac{1}{P + A_- + B_-} \tag{A9}$$
$$= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\exp(i\omega t + 0^+)}{\omega - R(\omega + \eta^A \Delta_A) - R(\omega + \eta^B \Delta_B) + i\gamma(\omega + \eta^A \Delta_A) + i\gamma(\omega + \eta^B \Delta_B)} d\omega$$
$$= \exp[i\omega_{12}^s t - \gamma(\omega_{12}^s + \eta^A \Delta_A) t - \gamma(\omega_{12}^s + \eta^B \Delta_B) t],$$

where ω_{12}^s is the solution of $\omega - R(\omega + \eta^A \Delta_A) - R(\omega + \eta^B \Delta_B) = 0$ and is the Lamb shift due to the two environments indirect interaction, which is a nonlocal effect. $\mathscr{L}^{-1} \frac{1}{P+A_++B_+}$

conjugates with $\mathscr{L}^{-1}\frac{1}{P+A_-+B_-}$.

$$\mathscr{L}^{-1}\frac{1}{P+A_++B_-} \tag{A10}$$
$$= \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\exp(i\omega t+0^+)}{\omega-R(\eta^A\Delta_A-\omega)-R(\omega+\eta^B\Delta_B)+i\gamma(\eta^A\Delta_A-\omega)+i\gamma(\omega+\eta^B\Delta_B)}d\omega$$
$$= \exp[i\omega_{12}^a t-\gamma(\eta^A\Delta_A-\omega_{12}^a)t-\gamma(\omega_{12}^a+\eta^B\Delta_B)t],$$

where ω_{12}^a is the solution of $\omega-R(\eta^A\Delta_A-\omega)-R(\omega+\eta^B\Delta_B)=0$ and is also the Lamb shift due to the two environment indirect interaction. In the same way, $\mathscr{L}^{-1}\frac{1}{P+A_-+B_+}$ conjugates with $\mathscr{L}^{-1}\frac{1}{P+A_++B_-}$.

$$\mathscr{L}^{-1}\frac{1}{P+A_++A_-+B_-} \tag{A11}$$
$$= \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\exp(i\omega t+0^+)/[\omega-R(\eta^A\Delta_A-\omega)-R(\eta^A\Delta_A+\omega)-R(\omega+\eta^B\Delta_B)$$
$$+i\gamma(\eta^A\Delta_A-\omega)+i\gamma(\eta^A\Delta_A+\omega)+i\gamma(\omega+\eta^B\Delta_B)]d\omega \tag{A12}$$
$$= \exp[i\omega_{31}t-\gamma(\eta^A\Delta_A-\omega_{31})t-\gamma(\eta^A\Delta_A+\omega_{31})t-\gamma(\omega_{31}+\eta^B\Delta_B)t],$$

where ω_{31} is the solution of $\omega-R(\eta^A\Delta_A-\omega)-R(\eta^A\Delta_A+\omega)-R(\omega+\eta^B\Delta_B)=0$ and is another Lamb shift due to nonlocal interaction. $\mathscr{L}^{-1}\frac{1}{P+A_++A_-+B_+}$ conjugates with $\mathscr{L}^{-1}\frac{1}{P+A_++A_-+B_-}$.

$$\mathscr{L}^{-1}\frac{1}{P+A_-+B_++B_-} \tag{A13}$$
$$= \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\exp(i\omega t+0^+)/[\omega-R(\omega+\eta^A\Delta_A)-R(\eta^B\Delta_B-\omega)-R(\eta^B\Delta_B+\omega)$$
$$+i\gamma(\omega+\eta^A\Delta_A)+i\gamma(\eta^B\Delta_B-\omega)+i\gamma(\eta^B\Delta_B+\omega)]d\omega$$
$$= \exp[i\omega_{32}t-\gamma(\omega_{32}+\eta^A\Delta_A)t-\gamma(\eta^B\Delta_B-\omega_{32})t-\gamma(\eta^B\Delta_B+\omega_{32})t],$$

where ω_{32} is the solution of $\omega-R(\omega+\eta^A\Delta_A)-R(\eta^B\Delta_B-\omega)-R(\eta^B\Delta_B+\omega)=0$ and is the Lamb shift due to the two environment indirect interaction, too. $\mathscr{L}^{-1}\frac{1}{P+A_-+B_++B_-}$

conjugates with $\mathscr{L}^{-1}\frac{1}{P+A_{+}+B_{+}+B_{-}}$.

$$\begin{aligned}
&\mathscr{L}^{-1}\frac{1}{P+A_{+}+A_{-}+B_{+}+B_{-}} \qquad \text{(A14)}\\
&= \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\exp(i\omega t+0^{+})}{\omega+i\gamma(\eta^{A}\Delta_{A}-\omega)+i\gamma(\eta^{A}\Delta_{A}+\omega)+i\gamma(\omega+\eta^{B}\Delta_{B})+i\gamma(\eta^{B}\Delta_{B}-\omega)}d\omega\\
&= \exp[-2\gamma(\eta^{A}\Delta_{A})t-2\gamma(\eta^{B}\Delta_{B})t].
\end{aligned}$$

[1] A. Barenco, A. Ekert, J. Mod. Opt, **42**, 1253 (1995).

[2] S. F. Pereira, Z. Y. Ou, H. J. Kimble, Phys. Rev. A. **62**, 042311 (2000).

[3] Artur K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).

[4] A. Barenco, Contemp. Phys. **37** (1996) 375.

[5] Lov K. Grover, Phys. Rev. Lett. **79** (1997) 325.

[6] J. J . Bollinger, Wayne M. Itano, D. J. Wineland and D. J. Heinzen, Phys. Rev. A **54** (1996) R4649.

[7] S. F. Huelga, C. Macchiavello, T. Pellizzari, A. K. Ekert, M. B. Plenio and J. I. Cirac, Phys. Rev. Lett. **79** (1997) 3865..

[8] M. Eibl, N. Kiesel, M. Bourennane, C. Kurtsiefer, H. Weinfurter, Phys. Rev. Lett. **92** (2004) 077901.

[9] O. Mandel, M. Greiner, A. Widera, T. Rom, T.W. Häsch, I. Bloch, Nature 425 (2003) 937.

[10] C.F. Roos, M. Riebe, H. Häfner, W. Häsel, J.B. Elm, G.P.T. Lancaster, C. Becher, F. Schmidt-Kaler, R. Blatt, Science **304** (2004) 1478. .

[11] Z. Zhao, Y.-A. Chen, A.-N. Zhang, T. Yang, H.J. Briegel, J.-W. Pan, Nature 430 (2004) 54..

[12] Y.A. Pashkin, T. Yamamoto, O. Astafiev, Y. Nakamura, D.V. Averin, J.S. Tsai, Nature 421 (2003) 823..

[13] A.J. Berkley, H. Xu, R.C. Ramos, M.A. Gubrud, F.W. Strauch, P.R. Johnson, J.R. Anderson, A.J. Dragt, C.J. Lobb, F.C. Wellstood, Science **300** (2003) 1548..

[14] C.F. Roos, G.P.T. Lancaster, M. Riebe, H. Häfner,W. Häsel, S. Gulde, C. Becher, J. Eschner, F. Schmidt-Kaler, R. Blatt, Phys. Rev. Lett. **92** (2004) 220402.

[15] M. P. Almeida, F. de Melo, M. Hor-Meyll, A. Salles, S. P. Walborn, P. H. Souto Ribeiro, L. Davidovich, Science **316** 579 (2007).

[16] Z. Ficek, R. Tanaś. Phys. Report **372** 369 (2002).

[17] FlorianMin tert, Andre R.R. Carvalho, Marek Kus and Andreas Buchleitner, Phys. Report **415** 207 (2005).

[18] T. Yu, J. H. Eberly Phys. Rev. lett. **97**, 140403 (2006).

[19] T. Yu, J. H. Eberly Phys. Rev. lett. **93**, 140404 (2004).

[20] Alexander V. Khaetskii, Daniel Loss, and Leonid Glazman, Phys. Rev. lett. **88**, 186802 (2002).

[21] H. Zheng, Eur. Phys. J. B **38**, 559 (2004); Zhuo-Jie Wu, Ka-Di Zhu, Xiao-Zhong Yuan, Yi-Wen Jiang, and Hang Zheng, Phys. Rev. B **71**, 205323 (2005); F. Guinea, V. Hakim, A. Muramatsu, Phys. Rev. B **32**, 4410 (1985).

[22] W. H. Louisell, *statistical properties of radiation*, Wiley, New York, (1973).

[23] Manzoor Ikram, Fu-li Li, and M. Suhail Zubairy, Phys. Rev. A **75** 062336 (2007)

[24] S. Bose, P. L. Knight, M. B. Plenio, and V. Vedral, Phys. Rev. lett **83**, 5158 (1999).

[25] G. Seewald, E. Hagn, and E. Zech, Phys. Rev. Lett **78** 5022 (1997).

[26] William K. Wootters, Phys. Rev. Lett **80** 2245 (1998).

[27] Gerald D. Mahan, *Many-Partical physics*, (World Scientific, New York, 1990).

Figure Caption

Fig. 1. The entanglement decay via spontaneous emission of two two-level qubits starting from the initially entangled state $(1-a)/3|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + a/3|\downarrow\downarrow\rangle\langle\downarrow\downarrow| + 1/3(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|)$ with a between zero and 1. the coupling constant of the environment and qubit $\alpha_A = \alpha_B = 0.01$. Here and in the following figures energies Δ_A and Δ_B are expressed in units of ω_c, times in units of ω_c^{-1}. We assume $\Delta_A = 0.2$, $\Delta_B = 0.4$.

Fig. 2. The entanglement decay via spontaneous emission of two two-level qubits the coupling constant of the environment and qubit $\alpha_A = \alpha_B = 0.05$. the other parameter the same as Fig. 1.

Fig. 3. The entanglement decay via spontaneous emission of two two-level qubits. the coupling constant of the environment and qubit $\alpha_A = \alpha_B = 0.1$. the other parameter the same as Fig. 1.

Fig.1

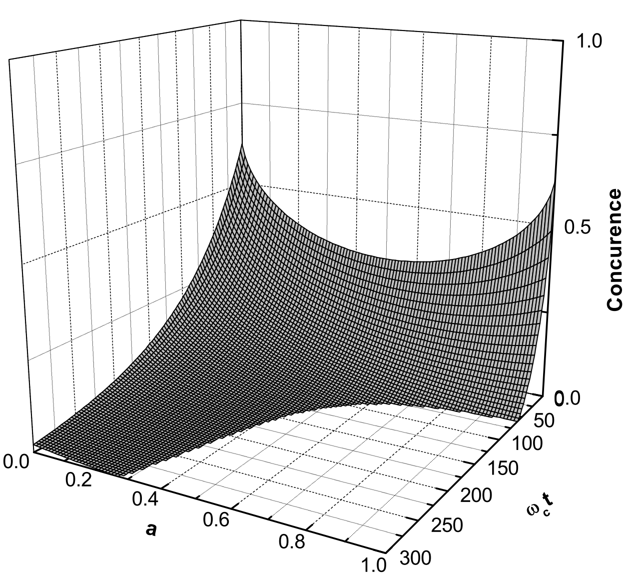
1.0
0.5
0.0
Concurence
0.0
0.2
0.4
0.6
0.8
1.0
a
50
100
150
200
250
300
$\omega_c t$

Fig.2

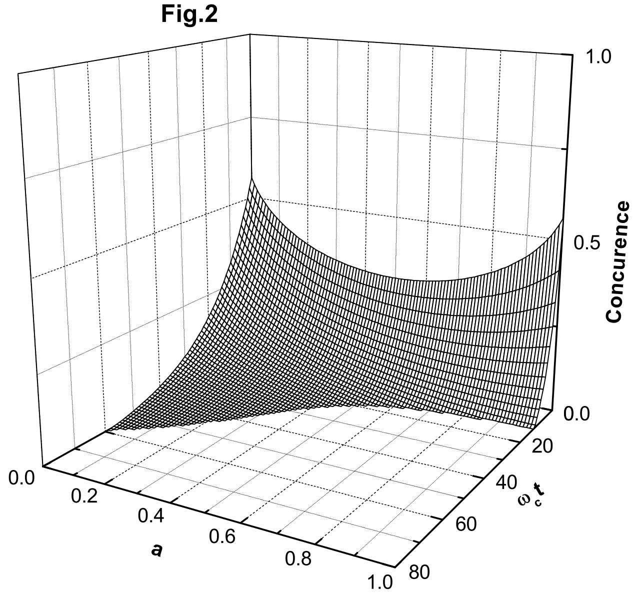

Fig.3